On Polynilpotent Multipliers of Free Nilpotent Groups*

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Abstract

In this paper, we present an explicit structure for the Baer invariant of a free nilpotent group (the n-th nilpotent product of the infinite cyclic group, $\mathbf{Z} \overset{n}{*} \mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}$) with respect to the variety of polynilpotent groups of class row $(c,1)$, $\mathcal{N}_{c,1}$, for all $c > 2n-2$. In particular, an explicit structure of the Baer invariant of a free abelian group with respect to the variety of metabelian groups will be presented.

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1.Introduction and Preliminaries

Historically, there have been several papers from the beginning of twentieth century, trying to find some structure for the well-known notion of the Schur multiplier and its varietal generalization, the Baer invariant, of some famous products of groups, such as direct products, free products, nilpotent products and regular products.

I. Schur [13] in 1907 and J. Wiegold [14] in 1971 obtained the structure of the Schur multiplier of the direct product of two finite groups as follows:

$M(A \times B) \cong M(A) \oplus M(B) \oplus \frac{[A,B]}{[A,B,A*B]}$, where $\frac{[A,B]}{[A,B,A*B]} \cong A_{ab} \otimes B_{ab}$.

In 1979, M. R. R. Moghaddam [11] and in 1998, G. Ellis [2], tried to generalize the above result to obtain the structure of the c-nilpotent multiplier of the direct product of two groups, $\mathcal{N}_c M(A \times B)$. Also in 1997 the second author in a joint paper [7] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group.

In 1972 W. Haebich [4] presented a formula for the Schur multiplier of a regular product of a family of groups. It is known that the regular product is a generalization of the nilpotent product and the last one is a generalization of the direct product, so Haebich's result is an interesting generalization of the Schur-Wiegold result. Also, M. R. R. Moghaddam [12], in 1979 gave a formula similar to Haebich's formula for the Schur multiplier of a nilpotent product. Moreover, in 1992, N. D. Gupta and M. R. R. Moghaddam [3] tried to present an explicit formula for the c-nilpotent multiplier of the n-th nilpotent product $\mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$ (see [8, Defn. 2.6] for the definition of the n-th nilpotent product).

In 2001, the second author [8] found a structure similar to Haebich's type for the c-nilpotent multiplier of a nilpotent product of a family of cyclic groups. The c-nilpotent multiplier of a free product of some cyclic groups was studied by the second author [9] in 2002.
Finally the authors [10] concentrated on the Baer invariant with respect to the variety of polynilpotent groups, for the first time. We succeeded to present an explicit formula for the polynilpotent multipliers of finitely generated abelian groups.
Now in this paper we intend to obtain an explicit formula for some polynilpotent multipliers of an n-th nilpotent product of some infinite cyclic groups,

$$\mathcal{N}_{c,1}M(\mathbf{Z} \overset{n}{*} \mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}), \text{ for all } c > 2n-2.$$

(Note that $\mathbf{Z} \overset{n}{*} \mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}$ has the presentation $F/\gamma_{n+1}(F)$, where F is a free group and hence it is a free nilpotent group of class n).

In particular, the structure of the metabelian multiplier of direct product of some infinite cyclic group, $\mathcal{S}_2M(\mathbf{Z}\oplus \mathbf{Z}\oplus \dots \oplus \mathbf{Z})$ is completely known.

In the following we present some preliminaries which are used in our method.

Definition 1.1. Let G be any group with a free presentation $G \cong F/R$, where F is a free group. Then, after R. Baer [1], the *Baer invariant* of G with respect to a variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} ,$$

where V is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F with respect to $\mathcal{V}$ and

$$[RV^*F] =< v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n)v(f_1, \dots, f_i, \dots, f_n)^{-1} \mid$$

$$r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbf{N} > .$$

In special case, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of G will be the well-known notion the *Schur multiplier*

$$\frac{R \cap F'}{[R, F]}.$$

If $\mathcal{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, then the Baer invariant of G with respect to $\mathcal{N}_c$ which is called the *c-nilpotent multiplier* of G, will be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,\ _1F] = [R, F], [R,\ _cF] = [[R,\ _{c-1}F], F]$, inductively.

In a very more general case, let $\mathcal{V}$ be the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$, $\mathcal{N}_{c_1,\ldots,c_t}$, then the Baer invariant of a group G with respect to this variety, which we call it a *polynilpotent multiplier*, is as follows:

$$\mathcal{N}_{c_1,\ldots,c_t} M(G) \cong \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R\mathcal{N}^*_{c_1,\ldots,c_t}F]}$$

$$\cong \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F), \ldots,\ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)]}, \quad (\star)$$

where $\gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))$ is a term of the iterated lower central series of F which is the verbal subgroup of F with respect to $\mathcal{N}_{c_1,\ldots,c_t}$. See [6, Corollary 6.14] for the equality

$$[R\mathcal{N}^*_{c_1,\ldots,c_t}F] = [R,\ _{c_1}F,\ _{c_2}\gamma_{c_1+1}(F), \ldots,\ _{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)].$$

Definition 1.2. *Basic commutators* are defined in the usual way. If X is a fully ordered independent subset of a free group, the basic commutators on X are defined inductively over their weight as follows:

(i) All the members of X are basic commutators of weight one on X;
(ii) Assuming that $n > 1$ and that the basic commutators of weight less than n on X have been defined and ordered;
(iii) A commutator $[a, b]$ is a basic commutator of weight n on X if $wt(a) + wt(b) = n$, $a < b$, and if $b = [b_1, b_2]$, then $b_2 \leq a$. The ordering of basic commutators is then extended to include those of weight n in any way such that those of weight less than n precede those of weight n. The natural way to define the order on basic commutators of the same weight is lexicographically, $[b_1, a_1] < [b_2, a_2]$ if $b_1 < b_2$ or if $b_1 = b_2$ and $a_1 < a_2$.

The next two theorems are vital in our investigation.

Theorem 1.3 (P.Hall [5]). *Let $F =< x_1, x_2, \ldots, x_d >$ be a free group, then*

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

Theorem 1.4 (Witt Formula [5]). *The number of basic commutators of weight n on d generators is given by the following formula:*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m} \quad ,$$

where $\mu(m)$ is the Mobious function, which is defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \ldots p_k^{\alpha_k} \quad \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \ldots p_s, \end{cases}$$

where the p_i, $1 \leq i \leq k$, are the distinct primes dividing m.

2 The Main Results

Let $G \cong \mathbf{Z} \overset{n}{*} \mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}$ (m-copies of $\mathbf{Z}$) be the n-th nilpotent product of m copies of the infinite cyclic group $\mathbf{Z}$. It is known that G is the free n-th nilpotent group of rank m or equivalently G is the free nilpotent group of class n, and so has the following free presentation

$$1 \longrightarrow \gamma_{n+1}(F) \longrightarrow F \longrightarrow G \longrightarrow 1,$$

where F is a free group on a set $X = \{x_1, x_2, \dots, x_m\}$.
Clearly the Baer invariant of G with respect to the variety of nilpotent groups of class at most $c \geq 1$, $\mathcal{N}_c$, is defined as follows

$$\mathcal{N}_c M(G) \cong \frac{\gamma_{n+1}(F) \cap \gamma_{c+1}(F)}{[\gamma_{n+1}(F), \ _c\ F]} = \frac{\gamma_{n+1}(F) \cap \gamma_{c+1}(F)}{\gamma_{n+c+1}(F)}.$$

In the following theorem, the structure of $\mathcal{N}_c M(G)$ will be presented.

Theorem 2.1. *Let G be a free n-th nilpotent group of rank m. Then*
(i) $\mathcal{N}_c M(G)$ is the free abelian group of rank $\sum_{i=c+1}^{c+n} \chi_i(m)$, for all $c \geq n$.
(ii)$\mathcal{N}_c M(G)$ is the free abelian group of rank $\sum_{i=n+1}^{c+n} \chi_i(m)$, for all $c \leq n$.

Proof. (i) Let $c \geq n$. Then

$$\mathcal{N}_c M(G) \cong \frac{\gamma_{c+1}(F)}{\gamma_{c+n+1}(F)}.$$

Now by P. Hall's Theorem (1.3) $\gamma_{c+1}(F)/\gamma_{c+n+1}(F)$ is a free abelian group freely generated by all basic commutators of weights $c+1, \dots, c+n$ on X. So its rank is $\sum_{i=c+1}^{c+n} \chi_i(m)$.
(ii) Since $c \leq n$, $\mathcal{N}_c M(G) \cong \gamma_{n+1}(F)/\gamma_{c+n+1}(F)$. Hence the result holds similar to i. $\square$

Now, we try to obtain the structure of some polynilpotent multipliers of G of the form

$$\mathcal{N}_{c,1}M(G),$$

where $c > 2n-2$. Using $(\star)$ we have

$$\mathcal{N}_{c,1}M(G) \cong \frac{\gamma_{n+1}(F) \cap \gamma_2(\gamma_{c+1}(F))}{[\gamma_{n+1}(F),\ _c\ F, \gamma_{c+1}(F)]} = \frac{\gamma_2(\gamma_{c+1}(F))}{[\gamma_{n+c+1}(F), \gamma_{c+1}(F)]}.$$

In order to find the structure of $\mathcal{N}_{c,1}M(G)$, we need the following notation and lemmas. Using Definition and Notation 1.2, put

Y= The set of all basic commutators on X of weights $c+1, \ldots, c+n$

Z= The set of all basic commutators on Y of weight 2.

Lemma 2.2. *With the above notation, if $c \geq n-1$, then every element of Z is a basic commutator on X.*

Proof. Every element of Z has the form $[b, a]$, where b and a belong to Y and $b > a$ *i.e.* b and a are basic commutators of weights $c+i$ and $c+j$ on X in which $1 \leq j \leq i \leq n$, respectively.

Now, let $b = [b_1, b_2]$. In order to show that $[b, a]$ is a basic commutator on X, it is enough to show that $b_2 \leq a$. Since $b = [b_1, b_2]$ is a basic commutator on X, so $b_1 > b_2$ and hence $wt(b_2) \leq \frac{1}{2} wt(b)$, since $b \in Y$, so $wt(b) \leq c+n$. Now, if $c \geq n-1$, then $\frac{1}{2}(c+n) < c+1$. Thus, we have

$$wt(b_2) \leq \frac{1}{2} wt(b) \leq \frac{1}{2}(c+n) < c+1 \leq c+i = wt(a).$$

Therefore $b_2 < a$ and hence the result holds. $\square$

Lemma 2.3. *With the above notation and assumption, if $c \geq n-1$, then we have*

$$\gamma_2(\gamma_{c+1}(F)) \equiv < Z > \quad mod \ \ [\gamma_{c+n+1}(F), \gamma_{c+1}(F)].$$

Proof. Let $[\alpha,\beta]$ be a generator of $\gamma_2(\gamma_{c+1}(F))$, so $\alpha,\beta \in \gamma_{c+1}(F)$. By P. Hall's Theorem (1.3) we can put $\alpha = \alpha_1\alpha_2\ldots\alpha_t\eta$ and $\beta = \beta_1\beta_2\ldots\beta_s\mu$, where $\alpha_1,\ldots,\alpha_t$ and $\beta_1,\ldots,\beta_s$ are basic commutators of weights $c+1,\ldots,c+n$ on X and η , $\mu \in \gamma_{c+n+1}(F)$.
By using commutator calculus, it is easy to see that

$$[\alpha,\beta] = \Pi_{i,j}[\alpha_i,\beta_j]^{f_{ij}}[\alpha_i,\mu]^{g_i}[\eta,\beta_j]^{h_j},$$

where $f_{ij}, g_i, h_j \in \gamma_{c+1}(F)$. Note that, since $c \geq n-1$, $wt([\alpha_i,\beta_j]) \geq 2c+2 \geq c+n+1$. Now, it is easy to see that

$$[\alpha_i,\mu],[\eta,\beta_j],[\alpha_i,\beta_j,f_{ij}] \in [\gamma_{c+n+1}(F),\gamma_{c+1}(F)], \text{ for all } 1 \neq f_{ij} \in \gamma_{c+1}(F).$$

Therefore we have

$$[\alpha,\beta] = \Pi_{i,j}[\alpha_i,\beta_j] \quad (mod[\gamma_{c+n+1}(F),\gamma_{c+1}(F)]).$$

It is easy to see that $\Pi_{i,j}[\alpha_i,\beta_j] \in < Z >$. Hence the result holds.□

Now, we define the following useful set

$$W = \{\ [b,a] \mid b \text{ and } a \text{ are basic commutators on } X \text{ such that } b > a,$$
$$wt(b) \geq c+n+1\ ,\ wt(a) \geq c+1\ ,\ wt(b)+wt(a) \leq 2c+2n+1\}.$$

Lemma 2.4. *With the above notation, if $c > 2n-2$, then every element of W is a basic commutator on X.*

Proof. Let $[b,a]$ be an element of W. By definition of W, b and a are basic commutators on X and $b > a$. Now, let $b = [b_1,b_2]$, where b_1 and b_2 are basic commutators on X. It is enough to show that $b_2 \leq a$. Since $b = [b_1,b_2]$ is a basic commutator, so $b_1 > b_2$ and hence $wt(b_2) \leq \frac{1}{2}wt(b)$. By definition of W, $c+1 \leq wt(a)$, $c+n+1 \leq wt(b) \leq 2c+2n+1-wt(a) \leq$

$2c+2n+1-(c+1)=c+2n$. Since $c>2n-2$ we have $\frac{1}{2}(c+2n)<c+1$. Thus $wt(b_2)\leq \frac{1}{2}wt(b)\leq \frac{1}{2}(c+2n)<c+1\leq wt(a)$. Hence the result holds.□

Lemma 2.5. *With the above notation and assumption, if $c\geq n-1$ then we have*

$$[\gamma_{c+n+1}(F),\gamma_{c+1}(F)]\equiv <W> \quad mod \;\; \gamma_{2c+2n+2}(F).$$

Proof. Let $[\alpha,\beta]$ be a generator of $[\gamma_{c+n+1}(F),\gamma c+1(F)]$, so $\alpha\in\gamma_{c+n+1}(F)$ and $\beta\in\gamma_{c+1}(F)$. By P. Hall's Theorem (1.3) and considering two free abelian groups $\gamma_{c+n+1}(F)/\gamma_{c+2n+1}(F)$ and $\gamma_{c+1}(F)/\gamma_{c+n+1}(F)$ we can write $\alpha=\alpha_1\alpha_2\ldots\alpha_t\eta$ and $\beta=\beta_1\beta_2\ldots\beta_s\mu$, where $\alpha_1,\ldots,\alpha_t$ are basic commutators of weights $c+n+1,\ldots,c+2n$ on X and $\eta\in\gamma_{c+2n+1}(F)$, and $\beta_1,\ldots,\beta_s$ are basic commutators of weights $c+1,\ldots,c+n$ on X and $\mu\in\gamma_{c+n+1}(F)$. By using commutator calculus, it is easy to see that

$$[\alpha,\beta]=\Pi_{i,j}[\alpha_i,\beta_j]^{f_{ij}}[\alpha_i,\mu]^{g_i}[\eta,\beta_j]^{h_j},$$

where $f_{ij},\;g_i,\;h_j\in\gamma_{c+1}(F)$. Now we have

$$wt(\alpha_i)+wt(\mu)\geq(c+n+1)+(c+n+1)\geq 2c+2n+2$$

$$wt(\eta)+wt(\beta_j)\geq(c+2n+1)+(c+1)\geq 2c+2n+2$$

$$wt(\alpha_i)+wt(\beta_j)+wt(f_{ij})\geq(c+n+1)+(c+1)+(c+1)=3c+n+3\geq 2c+2n+2$$
$$\text{for all } 1\neq f_{ij}\in\gamma_{c+1}(F), \text{ since } c\geq n-1.$$

Therefore

$$[\alpha,\beta]\equiv\Pi_{i,j}[\alpha_i,\beta_j] \;\; mod \;\; \gamma_{2c+2n+2}(F)$$

$$\equiv\Pi_{wt(\alpha_i)+wt(\beta_j)\leq 2c+2n+1}[\alpha_i,\beta_j] \;\; mod \;\; \gamma_{2c+2n+2}(F)\in W.$$

Hence the result holds.□

Now, we are in a position to state and proof the main result of the paper.

Theorem 2.6 *Let* $\boldsymbol{Z} \overset{n}{*} \boldsymbol{Z} \overset{n}{*} \dots \overset{n}{*} \boldsymbol{Z} \cong F/\gamma_{n+1}(F)$ *be the free nilpotent group of class* n*. With the above notation and assumption, if* $c > 2n-2$*, then* $\mathcal{N}_{c,1}M(\boldsymbol{Z} \overset{n}{*} \boldsymbol{Z} \overset{n}{*} \dots \overset{n}{*} \boldsymbol{Z})$ *is a free abelian group with the following basis*

$$B = \{\ z[\gamma_{c+n+1}(F), \gamma_{c+1}(F)]\ \mid\ z \in Z\ \}$$

(Note that if $c > 2n-2$*, since* $c \geq 1$*, then* $c \geq n-1$*).*

Proof. Clearly $\mathcal{N}_{c,1}M(\underbrace{\mathbf{Z} \overset{n}{*} \mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}}_{m-copies}) = \gamma_2(\gamma_{c+1}(F))/[\gamma_{c+n+1}(F), \gamma_{c+1}(F)]$ is an abelian group which is generated by B, using Lemma 2.3. So it is enough to show that elements of B are linearly independent. Suppose $\bar{z_1}, \dots, \bar{z_k} \in B$ and $\sum_{i=1}^{k} \alpha_i \bar{z_i} = 0$, where α_i's are integers. Clearly $\sum_{i=1}^{k} \alpha_i z_i \in [\gamma_{c+n+1}(F), \gamma_{c+1}(F)]$. $(\star\star)$

Now consider the abelian group $\gamma_{2c+2}(F)/\gamma_{2c+2n+2}(F)$, which is free abelian with the basis of all basic commutators on X of weights $2c+2, \dots, 2c+2n+1$, by P. Hall's Theorem (1.3). By Lemma 2.5 we have

$$\frac{\gamma_{2c+2n+2}(F)[\gamma_{c+n+1}(F), \gamma_{c+1}(F)]}{\gamma_{2c+2n+2}(F)} = <\ w\gamma_{2c+2n+2}(F)\ \mid\ w \in W\ > .$$

By $(\star\star)$, there exists $w_1, \dots, w_t \in W$, and integer numbers $\beta_1, \dots, \beta_t$ such that

$$\sum_{i=1}^{k} \alpha_i (z_i \gamma_{2c+2n+2}(F)) = \sum_{i=1}^{t} \beta_i (w_i \gamma_{2c+2n+2}(F)).$$

Therefore

$$\sum_{i=1}^{k} \alpha_i (z_i \gamma_{2c+2n+2}(F)) + \sum_{i=1}^{t} (-\beta_i)(w_i \gamma_{2c+2n+2}(F)) = 0. \quad (\star\star\star)$$

By Lemmas 2.2 and 2.4 every element of Z and W is a basic commutator on X of weights $2c+2, \dots, 2c+2n+1$. Also by considering the form of elements

of Z and W, it is easy to see that $Z \cap W = \phi$. Now, by $(\star\star\star)$ and considering the basis of the free abelian group $\gamma_{2c+2}(F)/\gamma_{2c+2n+2}(F)$ we have $\alpha_i = 0$ for all $1 \leq i \leq k$ and $\beta_i = 0$ for all $1 \leq i \leq t$. Hence the result holds.$\Box$

Corollary 2.7. *If $c > 2n-2$, then*
$\mathcal{N}_{c,1}M(\underbrace{\boldsymbol{Z} \overset{n}{*} \boldsymbol{Z} \overset{n}{*} \ldots \overset{n}{*} \boldsymbol{Z}}_{m-copies}) \cong \boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}$ *$(\chi_2(\chi_{c+1}(m)+\ldots+\chi_{c+n}(m))-$copies),*
where $\chi_j(i)$ is the number of all basic commutators of weight j on i letters.
In particular
$\mathcal{S}_2M(\underbrace{\boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}}_{m-copies}) \cong \boldsymbol{Z} \oplus \ldots \oplus \boldsymbol{Z}$ *$(\chi_2(\chi_2(m))$-copies),*
where $\mathcal{S}_2$ is the variety of all metabelian groups. (solvable groups of length at most 2). Note that the authors presented a similar structure for $\mathcal{S}_2M(\mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \oplus \mathbf{Z})$ in a different method. (See [10]).

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